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IDEALS OF THE FOURIER ALGEBRA, SUPPORTS AND HARMONIC OPERATORS

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ABSTRACT. We examine the common null spaces of families of Herz-Schur multipliers and apply our results to study jointly harmonic operators and their relation with jointly harmonic functionals. We show how an annihilation formula obtained in [1] can be used to give a short proof as well as a generalisation of a result of Neufang and Runde concerning harmonic operators with respect to a normalised positive definite function. We compare the two notions of support of an operator that have been studied in the literature and show how one can be expressed in terms of the other.

1. INTRODUCTION AND PRELIMINARIES

In this paper we investigate, for a locally compact group G , the common null spaces of families of Herz-Schur multipliers (or completely bounded multipliers of the Fourier algebra $A(G)$) and their relation to ideals of $A(G)$.

This provides a new perspective for our previous results in [1] concerning (weak* closed) spaces of operators on $L^2(G)$ which are simultaneously invariant under all Schur multipliers and under conjugation by the right regular representation of G on $L^2(G)$ (*jointly invariant* subspaces – see below for precise definitions).

At the same time, it provides a new approach to, as well as an extension of, a result of Neufang and Runde [17] concerning the space $\tilde{\mathcal{H}}_\sigma$ of operators which are ‘harmonic’ with respect to a positive definite normalised function $\sigma : G \rightarrow \mathbb{C}$.

If σ is a probability measure on a locally compact group G , a Borel function f on G satisfying the equation

$$f(x) = \int_G f(y^{-1}x) d\sigma(y)$$

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is called σ -harmonic (see [4]). There is a non-commutative analogue of harmonic functions in the context of Fourier algebras and group von Neumann algebras. The duality between the Fourier algebra $A(G)$ and $VN(G)$ may be considered as a non-commutative analogue of the duality between $L^1(G)$ and $L^\infty(G)$. If σ is a complex function in the Fourier-Stieltjes algebra $B(G)$, Chu and Lau introduce the space of σ -harmonic functionals on $A(G)$ as $\{T \in VN(G) : \sigma \cdot T = T\}$ [4]. This space is the annihilator of the norm closed ideal of $A(G)$ generated by the set $\{\sigma\phi - \phi : \phi \in A(G)\}$ and thus the study of harmonic functionals leads naturally to the study of the corresponding ideals of $A(G)$.

Another approach to harmonicity was introduced by Jaworski and Neufang in [10]. In [17] a connection between the two approaches is presented; the notion of σ -harmonic operators is introduced, using results of [18], as an extension of the notion of σ -harmonic functionals on $A(G)$ as defined and studied by Chu and Lau in [4]. One of the main results of Neufang and Runde is that $\tilde{\mathcal{H}}_\sigma$ is the von Neumann algebra on $L^2(G)$ generated by the algebra \mathcal{D} of multiplication operators together with the space \mathcal{H}_σ of harmonic functionals, considered as a subspace of the von Neumann algebra $VN(G)$ of the group.

It will be seen that this result can be obtained as a consequence of the fact (see Corollary 2.12) that, for any family Σ of completely bounded multipliers of $A(G)$, the space $\tilde{\mathcal{H}}_\Sigma$ of *jointly Σ -harmonic operators* can be obtained as the weak* closed \mathcal{D} -bimodule generated by the *jointly Σ -harmonic functionals* \mathcal{H}_Σ . In fact, the spaces $\tilde{\mathcal{H}}_\Sigma$ belong to the class of jointly invariant subspaces of $\mathcal{B}(L^2(G))$ studied in [1, Section 4].

The space \mathcal{H}_Σ is the annihilator in $VN(G)$ of a certain ideal of $A(G)$. Now from any given closed ideal J of the Fourier algebra $A(G)$, there are two ‘canonical’ ways to arrive at a weak* closed \mathcal{D} -bimodule of $\mathcal{B}(L^2(G))$. One way is to consider its annihilator J^\perp in $VN(G)$ and then take the weak* closed \mathcal{D} -bimodule generated by J^\perp . We call this bimodule $\text{Bim}(J^\perp)$. The other way is to take a suitable saturation $\text{Sat}(J)$ of J within the trace class operators on $L^2(G)$ (see Theorem 1.1), and then form its annihilator. This gives a masa bimodule $(\text{Sat } J)^\perp$ in $\mathcal{B}(L^2(G))$. In [1], we proved that these two procedures yield the same bimodule, that is, $\text{Bim}(J^\perp) = (\text{Sat } J)^\perp$. Our proof that $\tilde{\mathcal{H}}_\Sigma = \text{Bim}(\mathcal{H}_\Sigma)$ rests on this equality.

The notion of *support*, $\text{supp}_G(T)$, of an element $T \in VN(G)$ was introduced by Eymard in [5] by considering T as a linear functional on the function algebra $A(G)$; thus $\text{supp}_G(T)$ is a closed subset of G . This notion was extended by Neufang and Runde in [17] to an arbitrary $T \in \mathcal{B}(L^2(G))$ and used to describe harmonic operators. By considering joint supports, we show that this extended notion of G -support for an operator $T \in \mathcal{B}(L^2(G))$ coincides with the joint G -support of a family of elements of $VN(G)$ naturally associated to T (Proposition 3.5).

On the other hand, the notion of support of an operator T acting on $L^2(G)$ was first introduced by Arveson in [2] as a certain closed subset of $G \times G$. This notion was used in his study of what was later called operator synthesis. A different but related approach appears in [6], where the notion of ω -support, $\text{supp}_\omega(T)$, of T was introduced and used to establish a bijective correspondence between reflexive masa-bimodules and ω -closed subsets of $G \times G$.

We show that the joint G -support $\text{supp}_G(\mathcal{A})$ of an arbitrary family $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$ can be fully described in terms of its joint ω -support $\text{supp}_\omega(\mathcal{A})$ (Theorem 3.10). The converse does not hold in general, as the ω -support, being a subset of $G \times G$, contains in general more information about an arbitrary operator than its G -support (see Remark 3.12); however, in case \mathcal{A} is a (weak* closed) jointly invariant subspace, we show that its ω -support can be recovered from its G -support (Theorem 3.6). We also show that, if a set $\Omega \subseteq G \times G$ is invariant under all maps $(s, t) \rightarrow (sr, tr)$, $r \in G$, then Ω is marginally equivalent to an ω -closed set if and only if it is marginally equivalent to a (topologically) closed set. This can fail for non-invariant sets (see for example [6, p. 561]). For a related result, see [20, Proposition 7.3].

Preliminaries and Notation Throughout, G will denote a second countable locally compact group, equipped with left Haar measure. Denote by $\mathcal{D} \subseteq \mathcal{B}(L^2(G))$ the maximal abelian selfadjoint algebra (masa, for short) consisting of all multiplication operators $M_f : g \rightarrow fg$, where $f \in L^\infty(G)$. We write $\text{VN}(G)$ for the von Neumann algebra $\{\lambda_s : s \in G\}''$ generated by the left regular representation $s \rightarrow \lambda_s$ of G on $L^2(G)$ (here $(\lambda_s g)(t) = g(s^{-1}t)$).

Every element of the predual of $\text{VN}(G)$ is a vector functional, $\omega_{\xi, \eta} : T \rightarrow (T\xi, \eta)$, where $\xi, \eta \in L^2(G)$, and $\|\omega_{\xi, \eta}\|$ is the infimum of the products $\|\xi\|_2 \|\eta\|_2$ over all such representations. This predual can be identified [5] with the set $A(G)$ of all complex functions u on G of the form $s \rightarrow u(s) = \omega_{\xi, \eta}(\lambda_s)$. With the above norm and pointwise operations, $A(G)$ is a (commutative, regular, semi-simple) Banach algebra of continuous functions on G vanishing at infinity, called the *Fourier algebra* of G ; its Gelfand spectrum can be identified with G via point evaluations. The set $A_c(G)$ of compactly supported elements of $A(G)$ is dense in $A(G)$.

A function $\sigma : G \rightarrow \mathbb{C}$ is a *multiplier* of $A(G)$ if for all $u \in A(G)$ the pointwise product σu is again in $A(G)$. By duality, a multiplier σ induces a bounded operator $T \rightarrow \sigma \cdot T$ on $\text{VN}(G)$. We say σ is a *completely bounded* (or *Herz-Schur*) *multiplier*, and write $\sigma \in M^{\text{cb}}A(G)$, if the latter operator is completely bounded, that is, if there exists a constant K such that $\|[\sigma \cdot T_{ij}]\| \leq K \| [T_{ij}] \|$ for all $n \in \mathbb{N}$ and all $[T_{ij}] \in M_n(\text{VN}(G))$ (the latter being the space of all n by n matrices with entries in $\text{VN}(G)$). The least such constant is the *cb norm* of σ . The space $M^{\text{cb}}A(G)$ with pointwise operations and the cb norm is a Banach algebra into which $A(G)$ embeds contractively. For a subset $\Sigma \subseteq M^{\text{cb}}A(G)$, we let $Z(\Sigma) = \{s \in G : \sigma(s) = 0 \text{ for all } \sigma \in \Sigma\}$ be its *zero set*.

A subset $\Omega \subseteq G \times G$ is called *marginally null* if there exists a null set (with respect to Haar measure) $X \subseteq G$ such that $\Omega \subseteq (X \times G) \cup (G \times X)$. Two sets $\Omega, \Omega' \subseteq G \times G$ are *marginally equivalent* if their symmetric difference is a marginally null set; we write $\Omega_1 \cong \Omega_2$. A set $\Omega \subseteq G \times G$ is said to be *ω -open* if it is marginally equivalent to a *countable* union of Borel rectangles $A \times B$; it is called *ω -closed* when its complement is ω -open.

Given any set $\Omega \subseteq G \times G$, we denote by $\mathfrak{M}_{\max}(\Omega)$ the set of all $T \in \mathcal{B}(L^2(G))$ which are *supported* by Ω in the sense that $M_{\chi_B} T M_{\chi_A} = 0$ whenever $A \times B \subseteq G \times G$ is a Borel rectangle disjoint from Ω (we write χ_A for the characteristic function of a set A). Given any set $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ there exists a smallest, up to marginal equivalence, ω -closed set $\Omega \subseteq G \times G$ supporting every element of \mathcal{U} , *i.e.* such that $\mathcal{U} \subseteq \mathfrak{M}_{\max}(\Omega)$. This set is called *the ω -support* of \mathcal{U} and is denoted $\text{supp}_{\omega}(\mathcal{U})$ [6].

Two functions $h_1, h_2 : G \times G \rightarrow \mathbb{C}$ are said to be *marginally equivalent*, or equal *marginally almost everywhere (m.a.e.)*, if they differ on a marginally null set.

The predual of $\mathcal{B}(L^2(G))$ consists of all linear forms ω given by $\omega(T) := \sum_{i=1}^{\infty} (T f_i, g_i)$ where $f_i, g_i \in L^2(G)$ and $\sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2 < \infty$. Each such ω defines a trace class operator whose kernel is a function $h = h_{\omega} : G \times G \rightarrow \mathbb{C}$, unique up to marginal equivalence, given by $h(x, y) = \sum_{i=1}^{\infty} f_i(x) \bar{g}_i(y)$. This series converges marginally almost everywhere on $G \times G$. We use the notation $\langle T, h \rangle := \omega(T)$.

We write $T(G)$ for the Banach space of (marginal equivalence classes of) such functions, equipped with the norm of the predual of $\mathcal{B}(L^2(G))$.

Let $\mathfrak{S}(G)$ be the multiplier algebra of $T(G)$; by definition, a measurable function $w : G \times G \rightarrow \mathbb{C}$ belongs to $\mathfrak{S}(G)$ if the map $m_w : h \rightarrow wh$ leaves $T(G)$ invariant, that is, if wh is marginally equivalent to a function from $T(G)$, for every $h \in T(G)$. Note that the operator m_w is automatically bounded. The elements of $\mathfrak{S}(G)$ are called (*measurable*) *Schur multipliers*. By duality, every Schur multiplier induces a bounded operator S_w on $\mathcal{B}(L^2(G))$, given by

$$\langle S_w(T), h \rangle = \langle T, wh \rangle, \quad h \in T(G), \quad T \in \mathcal{B}(L^2(G)).$$

The operators of the form S_w , $w \in \mathfrak{S}(G)$, are precisely the bounded weak* continuous \mathcal{D} -bimodule maps on $\mathcal{B}(L^2(G))$ (see [7], [22], [19] and [15]).

A weak* closed subspace \mathcal{U} of $\mathcal{B}(L^2(G))$ is invariant under the maps S_w , $w \in \mathfrak{S}(G)$, if and only if it is invariant under all left and right multiplications by elements of \mathcal{D} , *i.e.* if $M_f T M_g \in \mathcal{U}$ for all $f, g \in L^{\infty}(G)$ and all $T \in \mathcal{U}$, in other words, if it is a *\mathcal{D} -bimodule*. For any set $\mathcal{T} \subseteq \mathcal{B}(L^2(G))$ we denote by $\text{Bim } \mathcal{T}$ the smallest weak* closed \mathcal{D} -bimodule containing \mathcal{T} ; thus, $\text{Bim}(\mathcal{T}) = \overline{[\mathfrak{S}(G)\mathcal{T}]^{w*}}$.

We call a subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ *invariant* if $\rho_r T \rho_r^* \in \mathcal{U}$ for all $T \in \mathcal{U}$ and all $r \in G$; here, $r \rightarrow \rho_r$ is the right regular representation of G on

$L^2(G)$. An invariant space, which is also a \mathcal{D} -bimodule, will be called a *jointly invariant space*.

It is not hard to see that, if $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$, the smallest weak* closed jointly invariant space containing \mathcal{A} is the weak* closed linear span of $\{S_w(\rho_r T \rho_r^*) : T \in \mathcal{A}, w \in \mathfrak{S}(G), r \in G\}$.

For a complex function u on G we let $N(u) : G \times G \rightarrow \mathbb{C}$ be the function given by $N(u)(s, t) = u(ts^{-1})$. For any subset E of G , we write $E^* = \{(s, t) \in G \times G : ts^{-1} \in E\}$.

It is shown in [3] (see also [11] and [23]) that the map $u \rightarrow N(u)$ is an isometry from $M^{\text{cb}}A(G)$ into $\mathfrak{S}(G)$ and that its range consists precisely of all *invariant* Schur multipliers, *i.e.* those $w \in \mathfrak{S}(G)$ for which $w(sr, tr) = w(s, t)$ for every $r \in G$ and marginally almost all s, t . Note that the corresponding operators $S_{N(u)}$ are denoted $\hat{\Theta}(u)$ in [18].

The following result from [1] is crucial for what follows.

Theorem 1.1. *Let $J \subseteq A(G)$ be a closed ideal and $\text{Sat}(J)$ be the closed $L^\infty(G)$ -bimodule of $T(G)$ (under the action $(f \cdot h \cdot g)(s, t) = f(s)h(s, t)g(t)$) where $f, g \in L^\infty(G)$ and $h \in T(G)$ generated by the set*

$$\{N(u)\chi_{L \times L} : u \in J, L \text{ compact}, L \subseteq G\}.$$

Then $\text{Sat}(J)^\perp = \text{Bim}(J^\perp)$.

2. NULL SPACES AND HARMONIC OPERATORS

Given a subset $\Sigma \subseteq M^{\text{cb}}A(G)$, let

$$\mathfrak{N}(\Sigma) = \{T \in \text{VN}(G) : \sigma \cdot T = 0, \text{ for all } \sigma \in \Sigma\}$$

be the *common null set* of the operators on $\text{VN}(G)$ of the form $T \rightarrow \sigma \cdot T$, with $\sigma \in \Sigma$. Letting

$$\Sigma A \stackrel{\text{def}}{=} \overline{\text{span}(\Sigma A(G))} = \overline{\text{span}\{\sigma u : \sigma \in \Sigma, u \in A(G)\}},$$

it is easy to verify that ΣA is a closed ideal of $A(G)$ and that

$$(1) \quad \mathfrak{N}(\Sigma) = (\Sigma A)^\perp.$$

Remark 2.1. *The sets of the form ΣA are precisely the closed ideals of $A(G)$ generated by their compactly supported elements.*

Proof. It is clear that, if $\Sigma \subseteq M^{\text{cb}}A(G)$, the set $\text{span}\{\sigma u : \sigma \in \Sigma, u \in A_c(G)\}$ consists of compactly supported elements and is dense in ΣA . Conversely, suppose that $J \subseteq A(G)$ is a closed ideal such that $J \cap A_c(G)$ is dense in J . For every $u \in J$ with compact support K , there exists $v \in A(G)$ which equals 1 on K [5, (3.2) Lemme], and so $u = uv \in JA$. Thus $J = \overline{J \cap A_c(G)} \subseteq JA \subseteq J$ and hence $J = JA$. \square

The following Proposition shows that it is sufficient to study sets of the form $\mathfrak{N}(J)$ where J is a closed ideal of $A(G)$.

Proposition 2.2. *For any subset Σ of $M^{\text{cb}}A(G)$,*

$$\mathfrak{N}(\Sigma) = \mathfrak{N}(\Sigma A).$$

Proof. If $\sigma \cdot T = 0$ for all $\sigma \in \Sigma$ then *a fortiori* $v\sigma \cdot T = 0$, for all $v \in A(G)$ and all $\sigma \in \Sigma$. It follows that $w \cdot T = 0$ for all $w \in \Sigma A$; thus $\mathfrak{N}(\Sigma) \subseteq \mathfrak{N}(\Sigma A)$.

Suppose conversely that $w \cdot T = 0$ for all $w \in \Sigma A$ and fix $\sigma \in \Sigma$. Now $u\sigma \cdot T = 0$ for all $u \in A(G)$, and so $\langle \sigma \cdot T, uv \rangle = 0$ when $u, v \in A(G)$. Since the products uv form a dense subset of $A(G)$, we have $\sigma \cdot T = 0$. Thus $\mathfrak{N}(\Sigma) \supseteq \mathfrak{N}(\Sigma A)$ since $\sigma \in \Sigma$ is arbitrary, and the proof is complete. \square

It is not hard to see that λ_s is in $\mathfrak{N}(\Sigma)$ if and only if s is in the zero set $Z(\Sigma)$ of Σ , and so $Z(\Sigma)$ coincides with the zero set of the ideal $J = \Sigma A$. Whether or not, for an ideal J , these unitaries suffice to generate $\mathfrak{N}(J)$ depends on properties of the zero set.

For our purposes, a closed subset $E \subseteq G$ is a *set of synthesis* if there is a unique closed ideal J of $A(G)$ with $Z(J) = E$. Note that this ideal is generated by its compactly supported elements [13, Theorem 5.1.6].

Lemma 2.3. *Let $J \subseteq A(G)$ be a closed ideal. Suppose that its zero set $E = Z(J)$ is a set of synthesis. Then*

$$\mathfrak{N}(J) = J^\perp = \overline{\text{span}\{\lambda_x : x \in E\}}^{w*}$$

Proof. Since E is a set of synthesis, $J = JA$ by Remark 2.1; thus $J^\perp = (JA)^\perp = \mathfrak{N}(J)$ by relation (1). The other equality is essentially a reformulation of the fact that E is a set of synthesis: a function $u \in A(G)$ is in J if and only if it vanishes at every point of E , that is, if and only if it annihilates every λ_s with $s \in E$ (since $\langle \lambda_s, u \rangle = u(s)$). \square

A linear space \mathcal{U} of bounded operators on a Hilbert space is called a *ternary ring of operators (TRO)* if it satisfies $ST^*R \in \mathcal{U}$ whenever S, T and R are in \mathcal{U} . Note that a TRO containing the identity operator is automatically a selfadjoint algebra.

Proposition 2.4. *Let $J \subseteq A(G)$ be a closed ideal. Suppose that its zero set $E = Z(J)$ is the coset of a closed subgroup of G . Then $\mathfrak{N}(J)$ is a (weak-* closed) TRO. In particular, if E is a closed subgroup then $\mathfrak{N}(J)$ is a von Neumann subalgebra of $\text{VN}(G)$.*

Proof. We may write $E = Hg$ where H is a closed subgroup and $g \in G$ (the proof for the case $E = gH$ is identical). Now E is a translate of H which is a set of synthesis by [24] and hence E is a set of synthesis. Thus Lemma 2.3 applies.

If sg, tg, rg are in E and $S = \lambda_{sg}$, $T = \lambda_{tg}$ and $R = \lambda_{rg}$, then $ST^*R = \lambda_{st^{-1}rg}$ is also in $\mathfrak{N}(J)$ because $st^{-1}rg \in E$. Since $\mathfrak{N}(J)$ is generated by $\{\lambda_x : x \in E\}$, it follows that $ST^*R \in \mathfrak{N}(J)$ for any three elements S, T, R of $\mathfrak{N}(J)$. \square

Remark 2.5. *Special cases of the above result are proved by Chu and Lau in [4] (see Propositions 3.2.10 and 3.3.9.)*

We now pass from $\text{VN}(G)$ to $\mathcal{B}(L^2(G))$: The algebra $M^{\text{cb}}A(G)$ acts on $\mathcal{B}(L^2(G))$ via the maps $S_{N(\sigma)}$, $\sigma \in M^{\text{cb}}A(G)$ (see [3] and [11]), and this action is an extension of the action of $M^{\text{cb}}A(G)$ on $\text{VN}(G)$: when $T \in \text{VN}(G)$ and $\sigma \in M^{\text{cb}}A(G)$, we have $S_{N(\sigma)}(T) = \sigma \cdot T$. Hence, letting

$$\tilde{\mathfrak{N}}(\Sigma) = \{T \in \mathcal{B}(L^2(G)) : S_{N(\sigma)}(T) = 0, \text{ for all } \sigma \in \Sigma\},$$

we have $\mathfrak{N}(\Sigma) = \tilde{\mathfrak{N}}(\Sigma) \cap \text{VN}(G)$.

The following is analogous to Proposition 2.2; note, however, that the dualities are different.

Proposition 2.6. *If $\Sigma \subseteq M^{\text{cb}}A(G)$,*

$$\tilde{\mathfrak{N}}(\Sigma) = \tilde{\mathfrak{N}}(\Sigma A).$$

Proof. The inclusion $\tilde{\mathfrak{N}}(\Sigma) \subseteq \tilde{\mathfrak{N}}(\Sigma A)$ follows as in the proof of Proposition 2.2. To prove that $\tilde{\mathfrak{N}}(\Sigma A) \subseteq \tilde{\mathfrak{N}}(\Sigma)$, let $T \in \tilde{\mathfrak{N}}(\Sigma A)$; then $S_{N(v\sigma)}(T) = 0$ for all $\sigma \in \Sigma$ and $v \in A(G)$. Thus, if $h \in T(G)$,

$$\langle S_{N(\sigma)}(T), N(v)h \rangle = \langle T, N(\sigma v)h \rangle = \langle S_{N(v\sigma)}(T), h \rangle = 0.$$

Since the linear span of the set $\{N(v)h : v \in A(G), h \in T(G)\}$ is dense in $T(G)$, it follows that $S_{N(\sigma)}(T) = 0$ and so $T \in \tilde{\mathfrak{N}}(\Sigma)$. \square

Proposition 2.7. *For every closed ideal J of $A(G)$, $\tilde{\mathfrak{N}}(J) = \text{Bim}(J^\perp)$.*

Proof. If $T \in \mathcal{B}(L^2(G))$, $h \in T(G)$ and $u \in A(G)$ then

$$\langle S_{N(u)}(T), h \rangle = \langle T, N(u)h \rangle.$$

By [1, Proposition 3.1], $\text{Sat}(J)$ is the closed linear span of $\{N(u)h : u \in J, h \in T(G)\}$. We conclude that $T \in (\text{Sat}(J))^\perp$ if and only if $S_{N(u)}(T) = 0$ for all $u \in J$, i.e. if and only if $T \in \tilde{\mathfrak{N}}(J)$. By Theorem 1.1, $(\text{Sat}(J))^\perp = \text{Bim}(J^\perp)$, and the proof is complete. \square

Theorem 2.8. *For any subset Σ of $M^{\text{cb}}A(G)$,*

$$\tilde{\mathfrak{N}}(\Sigma) = \text{Bim}(\mathfrak{N}(\Sigma)).$$

Proof. It follows from relation (1) that $\text{Bim}((\Sigma A)^\perp) = \text{Bim}(\mathfrak{N}(\Sigma))$. But $\text{Bim}((\Sigma A)^\perp) = \tilde{\mathfrak{N}}(\Sigma A)$ from Proposition 2.7 and $\tilde{\mathfrak{N}}(\Sigma A) = \tilde{\mathfrak{N}}(\Sigma)$ from Proposition 2.6. \square

More can be said when the zero set $Z(\Sigma)$ is a subgroup (or a coset) of G .

Lemma 2.9. *Let $J \subseteq A(G)$ be a closed ideal. Suppose that its zero set $E = Z(J)$ is a set of synthesis. Then*

$$(2) \quad \tilde{\mathfrak{N}}(J) = \overline{\text{span}\{M_g \lambda_x : x \in E, g \in L^\infty(G)\}}^{w*}$$

Proof. By Theorem 2.8, $\tilde{\mathfrak{N}}(J) = \text{Bim}(\mathfrak{N}(J))$ and thus, by Lemma 2.3, $\tilde{\mathfrak{N}}(J)$ is the weak* closed linear span of the monomials of the form $M_f \lambda_s M_g$ where $f, g \in L^\infty(G)$ and $s \in E$. But, because of the commutation relation $\lambda_s M_g = M_{g_s} \lambda_s$ (where $g_s(t) = g(s^{-1}t)$), we may write $M_f \lambda_s M_g = M_\phi \lambda_s$ where $\phi = fg_s \in L^\infty(G)$. \square

Theorem 2.10. *Let $J \subseteq A(G)$ be a closed ideal. Suppose that its zero set $E = Z(J)$ is the coset of a closed subgroup of G . Then $\tilde{\mathfrak{N}}(J)$ is a (weak* closed) TRO. In particular if E is a closed subgroup then $\tilde{\mathfrak{N}}(J)$ is a von Neumann subalgebra of $\mathcal{B}(L^2(G))$ and*

$$\tilde{\mathfrak{N}}(J) = (\mathcal{D} \cup \mathfrak{N}(J))'' = (\mathcal{D} \cup \{\lambda_x : x \in E\})''.$$

Proof. As in the proof of Proposition 2.4, we may take $E = Hg$. By Lemma 2.9, it suffices to check the TRO relation for monomials of the form $M_f \lambda_{sg}$; but, by the commutation relation, triple products $(M_f \lambda_{sg})(M_g \lambda_{tg})^*(M_h \lambda_{rg})$ of such monomials may be written in the form $M_\phi \lambda_{st^{-1}rg}$ and so belong to $\tilde{\mathfrak{N}}(J)$ when sg, tg and rg are in the coset E . Finally, when E is a closed subgroup, the last equalities follow from relation (2) and the bicommutant theorem. \square

We next extend the notions of σ -harmonic functionals [4] and operators [17] to jointly harmonic functionals and operators:

Definition 2.11. *Let $\Sigma \subseteq M^{\text{cb}}A(G)$. An element $T \in \text{VN}(G)$ will be called a Σ -harmonic functional if $\sigma \cdot T = T$ for all $\sigma \in \Sigma$. We write \mathcal{H}_Σ for the set of all Σ -harmonic functionals.*

An operator $T \in \mathcal{B}(L^2(G))$ will be called Σ -harmonic if $S_{N(\sigma)}(T) = T$ for all $\sigma \in \Sigma$. We write $\tilde{\mathcal{H}}_\Sigma$ for the set of all Σ -harmonic operators.

Explicitly, if $\Sigma' = \{\sigma - \mathbf{1} : \sigma \in \Sigma\}$,

$$\mathcal{H}_\Sigma = \{T \in \text{VN}(G) : \sigma \cdot T = T \text{ for all } \sigma \in \Sigma\} = \mathfrak{N}(\Sigma')$$

$$\text{and } \tilde{\mathcal{H}}_\Sigma = \{T \in \mathcal{B}(L^2(G)) : S_{N(\sigma)}(T) = T \text{ for all } \sigma \in \Sigma\} = \tilde{\mathfrak{N}}(\Sigma').$$

The following is an immediate consequence of Theorem 2.8.

Corollary 2.12. *Let $\Sigma \subseteq M^{\text{cb}}A(G)$. Then the weak* closed \mathcal{D} -bimodule $\text{Bim}(\mathcal{H}_\Sigma)$ generated by \mathcal{H}_Σ coincides with $\tilde{\mathcal{H}}_\Sigma$.*

Let σ be a positive definite normalised function and $\Sigma = \{\sigma\}$. In [17, Theorem 4.8], the authors prove, under some restrictions on G or σ (removed in [12]), that $\tilde{\mathcal{H}}_\Sigma$ coincides with the von Neumann algebra $(\mathcal{D} \cup \mathcal{H}_\Sigma)''$. We give a short proof of a more general result.

Denote by $P^1(G)$ the set of all positive definite normalised functions on G . Note that $P^1(G) \subseteq M^{\text{cb}}A(G)$.

Theorem 2.13. *Let $\Sigma \subseteq P^1(G)$. The space $\tilde{\mathcal{H}}_\Sigma$ is a von Neumann subalgebra of $\mathcal{B}(L^2(G))$, and $\tilde{\mathcal{H}}_\Sigma = (\mathcal{D} \cup \mathcal{H}_\Sigma)''$.*

Proof. Note that $\mathcal{H}_\Sigma = \mathfrak{N}(\Sigma') = \mathfrak{N}(\Sigma'A)$ and $\tilde{\mathcal{H}}_\Sigma = \tilde{\mathfrak{N}}(\Sigma') = \tilde{\mathfrak{N}}(\Sigma'A)$. Since $Z(\Sigma')$ is a closed subgroup [9, Proposition 32.6], it is a set of spectral synthesis [24]. Thus the result follows from Theorem 2.10. \square

Remark 2.14. *It is worth pointing out that $\tilde{\mathcal{H}}_\Sigma$ has an abelian commutant, since it contains a masa. In particular, it is a type I, and hence an injective, von Neumann algebra.*

In [1, Theorem 4.3] it was shown that a weak* closed subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ is jointly invariant if and only if it is of the form $\mathcal{U} = \text{Bim}(J^\perp)$ for a closed ideal $J \subseteq A(G)$. By Proposition 2.7, $\text{Bim}(J^\perp) = \tilde{\mathfrak{N}}(J)$, giving another equivalent description. In fact, the ideal J may be replaced by a subset of $M^{\text{cb}}A(G)$:

Proposition 2.15. *Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ be a weak* closed subspace. The following are equivalent:*

- (i) \mathcal{U} is jointly invariant;
- (ii) there exists a closed ideal $J \subseteq A(G)$ such that $\mathcal{U} = \tilde{\mathfrak{N}}(J)$;
- (iii) there exists a subset $\Sigma \subseteq M^{\text{cb}}A(G)$ such that $\mathcal{U} = \tilde{\mathfrak{N}}(\Sigma)$.

Proof. We observed the implication (i) \Rightarrow (ii) above, and (ii) \Rightarrow (iii) is trivial. Finally, (iii) \Rightarrow (i) follows from Theorem 2.8 and [1, Theorem 4.3]. \square

Remark 2.16. *It might also be observed that every weak* closed jointly invariant subspace \mathcal{U} is of the form $\mathcal{U} = \tilde{\mathcal{H}}_\Sigma$ for some $\Sigma \subseteq M^{\text{cb}}A(G)$.*

We end this section with a discussion on the ideals of the form ΣA : If J is a closed ideal of $A(G)$, then $JA \subseteq J$; thus, by (1) and Proposition 2.2, $J^\perp \subseteq \mathfrak{N}(J)$ and therefore $\text{Bim}(J^\perp) \subseteq \tilde{\mathfrak{N}}(J)$, since $\tilde{\mathfrak{N}}(J)$ is a \mathcal{D} -bimodule and contains $\mathfrak{N}(J)$. The equality $J^\perp = \mathfrak{N}(J)$ holds if and only if J is generated by its compactly supported elements, equivalently if $J = JA$ (see Remark 2.1). Indeed, by Proposition 2.2 we have $\mathfrak{N}(J) = \mathfrak{N}(JA) = (JA)^\perp$ and so the equality $J^\perp = \mathfrak{N}(J)$ is equivalent to $J^\perp = (JA)^\perp$. Interestingly, the inclusion $\text{Bim}(J^\perp) \subseteq \tilde{\mathfrak{N}}(J)$ is in fact always an equality (Proposition 2.7).

We do not know whether all closed ideals of $A(G)$ are of the form ΣA . They certainly are when $A(G)$ satisfies *Ditkin's condition at infinity* [13, Remark 5.1.8 (2)], namely if every $u \in A(G)$ is the limit of a sequence (uv_n) , with $v_n \in A_c(G)$. Since $A_c(G)$ is dense in $A(G)$, this is equivalent to the condition that every $u \in A(G)$ belongs to the closed ideal $uA(G)$.

This condition has been used before (see for example [14]). It certainly holds whenever $A(G)$ has a weak form of approximate identity; for instance, when G has the approximation property (AP) of Haagerup and Kraus [8] and a fortiori when G is amenable. See also the discussion in Remark 4.2 of [16] and the one following Corollary 4.7 of [1].

3. ANNIHILATORS AND SUPPORTS

The space $\mathcal{B}(L^2(G))$ acquires an $A(G)$ -module structure through the action given by $T \rightarrow S_{N(u)}(T)$ for $u \in A(G)$. In this section, given a set \mathcal{A} of operators on $L^2(G)$, we study the ideal of all $u \in A(G)$ which act trivially on \mathcal{A} ; this is the annihilator of \mathcal{A} for this action. Its zero set will be called the G -support of \mathcal{A} ; we relate this to the ω -support of \mathcal{A} defined in [6].

In [5], Eymard introduced, for $T \in \text{VN}(G)$, the ideal I_T of all $u \in A(G)$ satisfying $u \cdot T = 0$. We generalise this by defining, for a subset \mathcal{A} of $\mathcal{B}(L^2(G))$,

$$I_{\mathcal{A}} = \{u \in A(G) : S_{N(u)}(\mathcal{A}) = \{0\}\}.$$

It is easy to verify that $I_{\mathcal{A}}$ is a closed ideal of $A(G)$.

Let $\mathcal{U}(\mathcal{A})$ be the smallest weak* closed jointly invariant subspace containing \mathcal{A} . We next prove that $\mathcal{U}(\mathcal{A})$ coincides with the set $\tilde{\mathfrak{N}}(I_{\mathcal{A}})$ of all $T \in \mathcal{B}(L^2(G))$ satisfying $S_{N(u)}(T) = 0$ for all $u \in I_{\mathcal{A}}$.

Proposition 3.1. *Let $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$. If $\sigma \in M^{\text{cb}}A(G)$ then $S_{N(\sigma)}(\mathcal{A}) = \{0\}$ if and only if $S_{N(\sigma)}(\mathcal{U}(\mathcal{A})) = \{0\}$. Thus, $I_{\mathcal{A}} = I_{\mathcal{U}(\mathcal{A})}$.*

Proof. Recall that

$$\mathcal{U}(\mathcal{A}) = \overline{\text{span}\{S_w(\rho_r T \rho_r^*) : T \in \mathcal{A}, w \in \mathfrak{S}(G), r \in G\}}^{w*}.$$

The statement now follows immediately from the facts that $S_{N(\sigma)} \circ S_w = S_w \circ S_{N(\sigma)}$ for all $w \in \mathfrak{S}(G)$ and $S_{N(\sigma)} \circ \text{Ad}_{\rho_r} = \text{Ad}_{\rho_r} \circ S_{N(\sigma)}$ for all $r \in G$. The first commutation relation is obvious, and the second one can be seen as follows: Denoting by θ_r the predual of the map Ad_{ρ_r} , for all $h \in T(G)$ we have $\theta_r(N(\sigma)h) = N(\sigma)\theta_r(h)$ since $N(\sigma)$ is right invariant and so

$$\begin{aligned} \langle S_{N(\sigma)}(\rho_r T \rho_r^*), h \rangle &= \langle \rho_r T \rho_r^*, N(\sigma)h \rangle = \langle T, \theta_r(N(\sigma)h) \rangle \\ &= \langle T, N(\sigma)\theta_r(h) \rangle = \langle S_{N(\sigma)}(T), \theta_r(h) \rangle \\ &= \langle \rho_r(S_{N(\sigma)}(T))\rho_r^*, h \rangle. \end{aligned}$$

Thus $S_{N(\sigma)}(\rho_r T \rho_r^*) = \rho_r(S_{N(\sigma)}(T))\rho_r^*$. □

Theorem 3.2. *Let $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$. The bimodule $\tilde{\mathfrak{N}}(I_{\mathcal{A}})$ coincides with the smallest weak* closed jointly invariant subspace $\mathcal{U}(\mathcal{A})$ of $\mathcal{B}(L^2(G))$ containing \mathcal{A} .*

Proof. Since $\mathcal{U}(\mathcal{A})$ is weak* closed and jointly invariant, by [1, Theorem 4.3] it equals $\text{Bim}(J^\perp)$, where J is the closed ideal of $A(G)$ given by

$$J = \{u \in A(G) : N(u)\chi_{L \times L} \in (\mathcal{U}(\mathcal{A}))_\perp \text{ for all compact } L \subseteq G\}.$$

We show that $J \subseteq I_{\mathcal{A}}$. Suppose $u \in J$; then, for all $w \in \mathfrak{S}(G)$ and all $T \in \mathcal{A}$, since $S_w(T)$ is in $\mathcal{U}(\mathcal{A})$, by Theorem 1.1 it annihilates $N(u)\chi_{L \times L}$ for every compact $L \subseteq G$. It follows that

$$\langle S_{N(u)}(T), w\chi_{L \times L} \rangle = \langle T, N(u)w\chi_{L \times L} \rangle = \langle S_w(T), N(u)\chi_{L \times L} \rangle = 0$$

for all $w \in \mathfrak{S}(G)$ and all compact $L \subseteq G$. Taking $w = f \otimes \bar{g}$ with $f, g \in L^\infty(G)$ supported in L , this yields

$$(S_{N(u)}(T)f, g) = \langle S_{N(u)}(T), w\chi_{L \times L} \rangle = 0$$

for all compactly supported $f, g \in L^\infty(G)$ and therefore $S_{N(u)}(T) = 0$. Since this holds for all $T \in \mathcal{A}$, we have shown that $u \in I_{\mathcal{A}}$.

It follows that $\mathcal{U}(\mathcal{A}) = \text{Bim}(J^\perp) \supseteq \text{Bim}(I_{\mathcal{A}}^\perp)$. But $\text{Bim}(I_{\mathcal{A}}^\perp) = \tilde{\mathfrak{N}}(I_{\mathcal{A}})$ by Proposition 2.7, and this space is clearly jointly invariant and weak* closed. Since it contains \mathcal{A} , it also contains $\mathcal{U}(\mathcal{A})$ and so

$$\mathcal{U}(\mathcal{A}) = \text{Bim}(J^\perp) = \text{Bim}(I_{\mathcal{A}}^\perp) = \tilde{\mathfrak{N}}(I_{\mathcal{A}}). \quad \square$$

Supports of functionals and operators In [17], the authors generalise the notion of support of an element of $\text{VN}(G)$ introduced by Eymard [5] by defining, for an arbitrary $T \in \mathcal{B}(L^2(G))$,

$$\text{supp}_G T := \{x \in G : u(x) = 0 \text{ for all } u \in A(G) \text{ with } S_{N(u)}(T) = 0\}.$$

Notice that $\text{supp}_G T$ coincides with the zero set of the ideal I_T (see also [17, Proposition 3.3]). More generally, let us define the G -support of a subset \mathcal{A} of $\mathcal{B}(L^2(G))$ by

$$\text{supp}_G(\mathcal{A}) = Z(I_{\mathcal{A}}).$$

When $\mathcal{A} \subseteq \text{VN}(G)$, then $\text{supp}_G(\mathcal{A})$ is just the support of \mathcal{A} considered as a set of functionals on $A(G)$ as in [5].

The following is proved in [17] under the assumption that G has the approximation property of Haagerup and Kraus [8]:

Proposition 3.3. *Let $T \in \mathcal{B}(L^2(G))$. Then $\text{supp}_G(T) = \emptyset$ if and only if $T = 0$.*

Proof. It is clear that the empty set is the G -support of the zero operator. Conversely, suppose $\text{supp}_G(T) = \emptyset$, that is, $Z(I_T) = \emptyset$. This implies that $I_T = A(G)$ (see [5, Corollary 3.38]). Hence $S_{N(u)}(T) = 0$ for all $u \in A(G)$, and so for all $h \in T(G)$ we have

$$\langle T, N(u)h \rangle = \langle S_{N(u)}(T), h \rangle = 0.$$

Since the linear span of $\{N(u)h : u \in A(G), h \in T(G)\}$ is dense in $T(G)$, it follows that $T = 0$. \square

Proposition 3.4. *The G -support of a subset $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$ is the same as the G -support of the smallest weak* closed jointly invariant subspace $\mathcal{U}(\mathcal{A})$ containing \mathcal{A} .*

Proof. Since $I_{\mathcal{A}} = I_{\mathcal{U}(\mathcal{A})}$ (Proposition 3.1), this is immediate. \square

The following proposition shows that the G -support of a subset $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$ is in fact the support of a space of linear functionals on $A(G)$ (as used by Eymard): it can be obtained either by first forming the ideal $I_{\mathcal{A}}$ of all $u \in A(G)$ ‘annihilating’ \mathcal{A} (in the sense that $S_{N(u)}(\mathcal{A}) = \{0\}$) and then taking the support of the annihilator of $I_{\mathcal{A}}$ in $\text{VN}(G)$; alternatively, it can

be obtained by forming the smallest weak* closed jointly invariant subspace $\mathcal{U}(\mathcal{A})$ containing \mathcal{A} and then considering the support of the set of all the functionals on $A(G)$ which are contained in $\mathcal{U}(\mathcal{A})$.

Proposition 3.5. *The G -support of a subset $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$ coincides with the supports of the following spaces of functionals on $A(G)$:*

- (i) *the space $I_{\mathcal{A}}^{\perp} \subseteq \text{VN}(G)$*
- (ii) *the space $\mathcal{U}(\mathcal{A}) \cap \text{VN}(G) = \mathfrak{N}(I_{\mathcal{A}})$.*

Proof. By Proposition 2.7 and Theorem 3.2,

$$\mathcal{U}(\mathcal{A}) = \tilde{\mathfrak{N}}(I_{\mathcal{A}}) = \text{Bim}(I_{\mathcal{A}}^{\perp}).$$

Since the \mathcal{D} -bimodule $\text{Bim}(I_{\mathcal{A}}^{\perp})$ is jointly invariant, it coincides with $\mathcal{U}(I_{\mathcal{A}}^{\perp})$. Thus $\mathcal{U}(\mathcal{A}) = \mathcal{U}(I_{\mathcal{A}}^{\perp})$ and so Proposition 3.4 gives $\text{supp}_G(\mathcal{A}) = \text{supp}_G(I_{\mathcal{A}}^{\perp})$, proving part (i).

Note that $\mathcal{U}(\mathfrak{N}(I_{\mathcal{A}})) = \text{Bim}(\mathfrak{N}(I_{\mathcal{A}})) = \tilde{\mathfrak{N}}(I_{\mathcal{A}})$ and so $\mathcal{U}(\mathfrak{N}(I_{\mathcal{A}})) = \mathcal{U}(\mathcal{A})$. Thus by Proposition 3.4, $\mathfrak{N}(I_{\mathcal{A}})$ and \mathcal{A} have the same support. Since $\mathcal{U}(\mathcal{A}) \cap \text{VN}(G) = \tilde{\mathfrak{N}}(I_{\mathcal{A}}) \cap \text{VN}(G) = \mathfrak{N}(I_{\mathcal{A}})$, part (ii) follows. \square

We are now in a position to relate the G -support of a set of operators to their ω -support as introduced in [6].

Theorem 3.6. *Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ be a weak* closed jointly invariant subspace. Then*

$$\text{supp}_{\omega}(\mathcal{U}) \cong (\text{supp}_G(\mathcal{U}))^*.$$

In particular, the ω -support of a jointly invariant subspace is marginally equivalent to a topologically closed set.

Proof. Let $J = I_{\mathcal{U}}$. By definition, $\text{supp}_G(\mathcal{U}) = Z(J)$. By the proof of Theorem 3.2, $\mathcal{U} = \text{Bim}(J^{\perp})$, and hence, by Theorem 1.1, $\mathcal{U} = (\text{Sat } J)^{\perp}$. By [1, Section 5], $\text{supp}_{\omega}(\mathcal{U}) = \text{null}(\text{Sat } J) = (Z(J))^*$, where $\text{null}(\text{Sat } J)$ is the largest, up to marginal equivalence, ω -closed subset F of $G \times G$ such that $h|_F = 0$ for all $h \in \text{Sat } J$ (see [21]). The proof is complete. \square

Corollary 3.7. *Let $\Sigma \subseteq M^{\text{cb}}A(G)$. Then*

$$\text{supp}_{\omega} \tilde{\mathfrak{N}}(\Sigma) \cong Z(\Sigma)^*.$$

If $Z(\Sigma)$ satisfies spectral synthesis, then $\tilde{\mathfrak{N}}(\Sigma) = \mathfrak{M}_{\max}(Z(\Sigma)^)$.*

Proof. From Theorem 2.8, we know that $\tilde{\mathfrak{N}}(\Sigma) = \text{Bim}((\Sigma A)^{\perp}) = \tilde{\mathfrak{N}}(\Sigma A)$ and so $\text{supp}_{\omega} \tilde{\mathfrak{N}}(\Sigma) \cong Z(\Sigma A)^*$ by [1, Section 5]. But $Z(\Sigma A) = Z(\Sigma)$ as can easily be verified (if $\sigma(t) \neq 0$ there exists $u \in A(G)$ so that $(\sigma u)(t) \neq 0$; the converse is trivial).

The last claim follows from the fact that, when $Z(\Sigma)$ satisfies spectral synthesis, there is a unique weak* closed \mathcal{D} bimodule whose ω -support is $Z(\Sigma)^*$ (see [16, Theorem 4.11] or the proof of [1, Theorem 5.5]). \square

Note that when $\Sigma \subseteq P^1(G)$, the set $Z(\Sigma)$ satisfies spectral synthesis.

The following corollary is a direct consequence of Corollary 3.7.

Corollary 3.8. *Let $\Sigma \subseteq M^{\text{cb}}A(G)$ and $\Sigma' = \{1 - \sigma : \sigma \in \Sigma\}$. If $Z(\Sigma')$ is a set of spectral synthesis, then $\tilde{\mathcal{H}}_\Sigma = \mathfrak{M}_{\max}(Z(\Sigma')^*)$.*

Corollary 3.9. *Let Ω be a subset of $G \times G$ which is invariant under all maps $(s, t) \rightarrow (sr, tr)$, $r \in G$. Then Ω is marginally equivalent to an ω -closed set if and only if it is marginally equivalent to a topologically closed set.*

Proof. A topologically closed set is of course ω -closed. For the converse, let $\mathcal{U} = \mathfrak{M}_{\max}(\Omega)$, so that $\Omega \cong \text{supp}_\omega(\mathcal{U})$. Note that \mathcal{U} is a weak* closed jointly invariant space. Indeed, since Ω is invariant, for every $T \in \mathcal{U}$ the operator $T_r =: \rho_r T \rho_r^*$ is supported in Ω and hence is in \mathcal{U} . Of course \mathcal{U} is invariant under all Schur multipliers. By Theorem 3.6, $\text{supp}_\omega(\mathcal{U})$ is marginally equivalent to a closed set. \square

Theorem 3.10. *Let $\mathcal{A} \subseteq \mathcal{B}(L^2(G))$. Then $\text{supp}_G(\mathcal{A})$ is the smallest closed subset $E \subseteq G$ such that E^* marginally contains $\text{supp}_\omega(\mathcal{A})$.*

Proof. Let $\mathcal{U} = \mathcal{U}(\mathcal{A})$ be the smallest jointly invariant weak* closed subspace containing \mathcal{A} . Let $Z = Z(I_{\mathcal{A}})$; by definition, $Z = \text{supp}_G \mathcal{A}$. But $\text{supp}_G \mathcal{A} = \text{supp}_G \mathcal{U} = Z$ (Proposition 3.4) and so $\text{supp}_\omega \mathcal{U} \cong Z^*$ by Theorem 3.6.

Thus Z^* does marginally contain $\text{supp}_\omega(\mathcal{A})$.

On the other hand, let $E \subseteq G$ be a closed set such that E^* marginally contains $\text{supp}_\omega(\mathcal{A})$. Thus any operator $T \in \mathcal{A}$ is supported in E^* . But since E^* is invariant, $\rho_r T \rho_r^*$ is also supported in E^* , for every $r \in G$. Thus \mathcal{U} is supported in E^* .

This means that Z^* is marginally contained in E^* ; that is, there is a null set $N \subseteq G$ such that $Z^* \setminus E^* \subseteq (N \times G) \cup (G \times N)$. We claim that $Z \subseteq E$. To see this, assume, by way of contradiction, that there exists $s \in Z \setminus E$. Then the ‘diagonal’ $\{(r, sr) : r \in G\}$ is a subset of $Z^* \setminus E^* \subseteq (N \times G) \cup (G \times N)$. It follows that for every $r \in G$, either $r \in N$ or $sr \in N$, which means that $r \in s^{-1}N$. Hence $G \subseteq N \cup s^{-1}N$, which is a null set. This contradiction shows that $Z \subseteq E$. \square

We note that for subsets \mathcal{S} of $\text{VN}(G)$ the relation $\text{supp}_\omega(\mathcal{S}) \subseteq (\text{supp}_G(\mathcal{S}))^*$ is in [16, Lemma 4.1].

In [17] the authors define, for a closed subset Z of G , the set

$$\mathcal{B}_Z(L^2(G)) = \{T \in \mathcal{B}(L^2(G)) : \text{supp}_G(T) \subseteq Z\}.$$

Corollary 3.11. *If $Z \subseteq G$ is closed, the set $\mathcal{B}_Z(L^2(G))$ consists of all $T \in \mathcal{B}(L^2(G))$ which are ω -supported in Z^* ; that is, $\mathcal{B}_Z(L^2(G)) = \mathfrak{M}_{\max}(Z^*)$. In particular, this space is a reflexive jointly invariant subspace.*

Proof. If T is ω -supported in Z^* , then by Theorem 3.10, $\text{supp}_G(T) \subseteq Z$.

Conversely if $\text{supp}_G(T) \subseteq Z$ then $\text{supp}_G(\mathcal{U}(T)) \subseteq Z$ by Proposition 3.4. But, by Theorem 3.6, $\text{supp}_\omega(\mathcal{U}(T)) \cong (\text{supp}_G(\mathcal{U}(T)))^* \subseteq Z^*$ and so T is ω -supported in Z^* . \square

Remark 3.12. The ω -support $\text{supp}_\omega(\mathcal{A})$ of a set \mathcal{A} of operators is more ‘sensitive’ than $\text{supp}_G(\mathcal{A})$ in that it encodes more information about \mathcal{A} . Indeed, $\text{supp}_G(\mathcal{A})$ only depends on the (weak* closed) jointly invariant subspace generated by \mathcal{A} , while $\text{supp}_\omega(\mathcal{A})$ depends on the (weak* closed) masa-bimodule generated by \mathcal{A} .

Example 3.13. Let $G = \mathbb{Z}$ and $\mathcal{A} = \mathfrak{M}_{\max}\{(i, j) : i + j \in \{0, 1\}\}$. The ω -support of \mathcal{A} is of course the two-line set $\{(i, j) : i + j \in \{0, 1\}\}$, while its G -support is \mathbb{Z} which gives no information about \mathcal{A} .

Indeed, if $E \subseteq \mathbb{Z}$ contains $\text{supp}_G(\mathcal{A})$, then by Theorem 3.10 $E^* = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : m - n \in E\}$ must contain $\{(i, j) : i + j \in \{0, 1\}\}$. Thus for all $n \in \mathbb{Z}$, since $(-n, n)$ and $(-n, n + 1)$ are in $\text{supp}_\omega(\mathcal{A})$ we have $n - (-n) \in E$ and $n + 1 - (-n) \in E$; hence $\mathbb{Z} \subseteq E$.

Example 3.14. Let G be a second countable locally compact group,

$$\tilde{\Delta} = \{(s, s^{-1}) : s \in G\}$$

be its antidiagonal and $\mathcal{A} = \mathfrak{M}_{\max}(\tilde{\Delta})$. Set $G^2 = \{s^2 : s \in G\}$. Then $\text{supp}_G(\mathcal{A}) = \overline{G^2}$.

Indeed, since $\tilde{\Delta}$ is ω -closed, the ω -support of \mathcal{A} coincides with $\tilde{\Delta}$. Suppose that $E \subseteq G$ is a closed subset such that E^* marginally contains $\tilde{\Delta}$. Then there exists a null set $M \subseteq G$ such that

$$\{(s^{-1}, s) : s \in M^c\} \subseteq E^*,$$

and hence $\{s^2 : s \in M^c\} \subseteq E$. Since M^c is dense and E is closed, the latter inclusion implies that $G^2 \subseteq E$ and hence $\overline{G^2} \subseteq E$. Theorem 3.10 now establishes the claim.

Note that, if $G = \mathbb{R}$ then $G^2 = \mathbb{R}$ and hence $\text{supp}_G(\mathcal{A}) = \mathbb{R}$.

Example 3.15. Let G be a second countable locally compact group and χ_E be the characteristic function of a Borel set $E \subseteq G$. It is easy to see that the ω -support of the multiplication operator M_a is equal (up to marginal equivalence) to $\{(s, s) : s \in E\}$. On the other hand, $\text{supp}_G(M_a) = \{e\}$. This can be seen either by direct verification (see [17], p. 678), or by an application of Theorem 3.10.

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